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# Association of Normal Random Variables and Slepian's Inequality

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Association of Normal Variables

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## I. Introduction

We give a simple proof of the result in Pitt (1981) that positively correlated normal random variables are associated. The proof is an adaptation of the original proof of Slepian's inequality in Slepian (1962), and extends to the case of elliptically contoured distributions.

## II. Normal Random Vectors

Let $X = (x_1,\ldots,x_n)$ be a mean zero n-dimensional normal random vector with n×n covariance matrix $\Sigma = (\sigma_{ij})$. By a smooth function we will mean a $C^2$ function $h(x)$ which together with its first and second order derivatives satisfy a $O(|x|^N)$ growth condition at $\infty$, for some finite N.

We set

$$H(\Sigma) = Eh(X),$$

and we are interested in the manner that $H(\Sigma)$ varies with $\Sigma$. Our main result is

Proposition 1: Let $\Gamma$ be another covariance matrix with $\gamma_{ii} = \sigma_{ii}$ and $\gamma_{ij} \leq \sigma_{ij}$ for all i and j. If h is a smooth function on $R^n$ and if

$$(1) \qquad \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \geq 0 \quad \text{for all } i \text{ and } j \text{ with } \gamma_{ij} < \sigma_{ij},$$

then

$$(2) \qquad H(\Gamma) \leq H(\Sigma).$$

Proof: By standard approximation arguments it suffices to show

$$\frac{\partial H(\Sigma)}{\partial \sigma_{ij}} \geq 0$$

whenever $\Sigma$ is nonsingular and $\partial^2 h/\partial x_i \partial x_j \geq 0$. Let $\phi(x) = \phi_\Sigma(x)$ be the mean zero normal density on $R^n$ with covariance matrix $\Sigma$. Then

$$(3) \qquad \frac{\partial\phi}{\partial\sigma_{ii}} = \frac{1}{2}\frac{\partial^2\phi}{\partial x_i^2}, \qquad \frac{\partial\phi}{\partial\sigma_{ij}} = \frac{\partial^2\phi}{\partial x_i \partial x_j}, \qquad i \neq j.$$

See e.g. Plackett (1954). Using (3) and our assumptions on h which justify two integrations by parts we have

$$(4) \qquad \frac{\partial H(\Sigma)}{\partial\sigma_{ij}} = \int_{R^n} h(x)\, \frac{\partial\phi(x)}{\partial\sigma_{ij}}\, dx$$

$$= \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j}\, \phi(x)\, dx$$

$$\geq 0,$$

which completes the proof.

By varying h, $\Gamma$ and $\Sigma$ we obtain other results.

Corollary 1. Let $h(x_1,\ldots,x_n) = f(x_1,\ldots,x_k)g(x_{k+1},\ldots,x_n)$ where f and g are bounded measurable increasing functions. Suppose also that

$$\gamma_{ii} = \sigma_{ii} \qquad \text{for all } i,$$

$$\gamma_{ij} = \sigma_{ij} \qquad \text{if } 1 \leq i, j \leq k \text{ or } k < i, j \leq n,$$

$$\gamma_{ij} \leq \sigma_{ij} \qquad \text{if } 1 \leq i \leq k < j \leq n.$$

Then

$$H(\Gamma) \leq H(\Sigma).$$

In particular, if $\sigma_{ij} \geq 0$ for $1 \leq i \leq k < j \leq n$ .

$$Ef(x_1,\ldots,x_k)Eg(x_{k+1},\ldots,x_n) \leq Ef(x_1,\ldots,x_k)g(x_{k+1},\ldots,x_n).$$

Proof: If f and g are smooth then h is smooth and

$$\frac{\partial^2 h}{\partial x_i \partial x_j} = \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} \geq 0 \quad \text{for } 1 \leq i \leq k < j \leq n.$$

In this case the result follows from (2). The general case follows by approximations as in Pitt (1981).

To state Slepian's inequality, we write $P_\Sigma(A)$ for the probability of the event that $X \in A \subset R^n$.

Corollary 2. (Slepian) If $\gamma_{ii} = \sigma_{ii}$ and $\gamma_{ij} \leq \sigma_{ij}$ for all i and j then for each $\lambda$

$$P_\Gamma\{\max X_i \leq \lambda\} \leq P_\Sigma\{\max X_i \leq \lambda\}.$$

Proof: Again by standard arguments it suffices to show that $H(\Gamma) \leq H(\Sigma)$ for each product $h(x) = \prod_1^n f_i(x_i)$ of bounded non-negative smooth decreasing function $f_i(x_i)$. For $i \neq j$, each such product satisfies $\partial^2 h/\partial x_i \partial x_j \geq 0$, and the result follows from Proposition 1.

Remark: Several variants are possible. In particular, changing the sign of $x_1,\dots,x_k$ in Corollary 1 gives the result of Jogdeo and Proschan (1981): If $\sigma_{ij} \le 0$ for $1 \le i \le k < j < n$, and if f and g are increasing then

$$Ef(x_1,\dots,x_k)g(x_{k+1},\dots,x_n) \le Ef(x_1,\dots,x_k)Eg(x_{k+1},\dots,x_n).$$

III. Elliptically Contoured Distributions

The previous results extend to elliptically contoured distributions. The extension of Slepian's inequality to this case was given in Das Gupta, Eaton, Olkin, Perlman, Savage and Sobel (1972).

Let $\langle x,y\rangle$ denote the Euclidean inner product on $R^n$ and let $\Sigma$ be a non-singular positive definite matrix. A probability density on $R^n$ of the form

$$p_\Sigma(x) = |\Sigma|^{-\frac{1}{2}}\, p(\langle x,\Sigma^{-1}x\rangle)$$

is called elliptically contoured. Here $p(\lambda) \ge 0$ is defined on $[0,\infty)$ and is assumed to satisfy $\int_0^\infty \lambda^{n-1}p(\lambda)d\lambda < \infty$.

We write

$$H(\Sigma) = \int_{R^n} h(x)p_\Sigma(x)\,dx,$$

and with this notation we will show that Proposition 1 remains valid.

It will suffice to establish (2) under the technical condition that $p(\lambda)$ is a $C^2$ function with compact support, and under this hypothesis we will show that:

(5) If $\partial^2h/\partial x_i\partial x_j \ge 0$, then $\frac{\partial}{\partial\sigma_{ij}} H(\Sigma) \ge 0$.

The proof is similar to the earlier one, but requires a substitute for the equations (3). This is supplied by Proposition 2.

For $\lambda \geq 0$ we set

$$F(\lambda) = \int_0^\lambda p(\xi)\, d\xi .$$

Let

$$F_\infty = \int_0^\infty p(\xi)\, d\xi,$$

and

$$G_\Sigma(x) = (2|\Sigma|)^{-\frac{1}{2}} \{F_\infty - F(\langle x, \Sigma^{-1}x\rangle)\}.$$

Proposition 2. If $h(x)$ is smooth and $p(\lambda)$ is $C^2$ with compact support, then

$$(6) \qquad \frac{\partial}{\partial\sigma_{ij}} \int_{R^n} h(x) p_\Sigma(x)\, dx = \int_{R^n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} G_\Sigma(x)\, dx.$$

Remarks: Since $G_\Sigma \geq 0$ this proves (5). Also, in the case that $p_\Sigma = \phi_\Sigma$ is a normal density one easily checks that $G_\Sigma = \phi_\Sigma$, so (6) is a generalization of (3).

Proof: Let $\Sigma^{-1} = (\sigma^{ij})$. We will use the matrix identities

$$(7) \qquad \frac{\partial}{\partial\sigma_{ii}} |\Sigma|^{-\frac{1}{2}} = -\tfrac{1}{2}\sigma^{ii}|\Sigma|^{-\frac{1}{2}},$$

$$\frac{\partial}{\partial\sigma_{ij}} |\Sigma|^{-\frac{1}{2}} = -\sigma^{ij}|\Sigma|^{-\frac{1}{2}}, \qquad i \neq j$$

$$\frac{\partial}{\partial\sigma_{ii}} \langle x, \Sigma^{-1}x\rangle = -\Big(\sum_{k=1}^n \sigma^{ik}x_k\Big)^2,$$

$$\frac{\partial}{\partial\sigma_{ij}} \langle x, \Sigma^{-1}x\rangle = -2\Big(\sum_{k=1}^n \sigma^{ik}x_k\Big)\Big(\sum_{\ell=1}^n \sigma^{j\ell}x_\ell\Big), \quad i \neq j,$$

without further comment.

Calculating, we now have

$$\frac{\partial p_\Sigma}{\partial \sigma_{ij}} = -\sigma^{ij}\, p_\Sigma - 2|\Sigma|^{-\frac{1}{2}}\, p'(\langle x,\Sigma^{-1}x\rangle)\Big(\sum_{k=1}^{n} \sigma^{ik}x_k\Big)\Big(\sum_{\ell=1}^{n} \sigma^{j\ell}x_\ell\Big)$$

$$= -\sigma^{ij}\, p_\Sigma - \Big(\sum_{k=1}^{n}\sigma^{ik}x_k\Big)\frac{\partial p_\Sigma}{\partial x_j} \;.$$

Our assumptions on p justify integration by parts and we have

$$\frac{\partial}{\partial \sigma_{ij}}\int_{R^n} h(x)p_\Sigma(x)dx = -\sigma^{ij}\int_{R^n} h(x)p_\Sigma(x)dx + \int_{R^n} \frac{\partial}{\partial x_j}\Big[\Big(\sum_{k=1}^{n}\sigma^{ik}x_k\Big)h(x)\Big]p_\Sigma(x)dx$$

$$= \int_{R^n} \Big(\sum_{k=1}^{n} \sigma^{ik}x_k\Big)\frac{\partial h(x)}{\partial x_j}\, p_\Sigma(x)dx$$

$$= \frac{1}{2|\Sigma|^{\frac{1}{2}}}\int_{R^n} \frac{\partial h(x)}{\partial x_j}\,\frac{\partial}{\partial x_i} F(\langle x,\Sigma^{-1}x\rangle)dx$$

$$= \int_{R_n} \frac{\partial^2 h(x)}{\partial x_i \partial x_j}\, G_\Sigma(x)dx.$$

The above calculations allow one extension which is perhaps worthwhile making. Using (7) we can easily verify that

$$(8) \qquad \frac{\partial}{\partial \sigma_{ii}}\int_{R^n} h(x)p_\Sigma(x)dx = \frac{1}{2}\int_{R^n} \frac{\partial^2 h(x)}{\partial x_i^2}\, G_\Sigma(x)dx.$$

Combining (6) and (8) it is elementary to prove

Proposition 3. Let $\Gamma$ and $\Sigma$ be positive definite matrices and set $A = \Sigma - \Gamma = (a_{ij})$. Let $\Sigma_t = \Gamma + tA$ and let $h(x)$ be a smooth function satisfying

$$(9) \qquad Ah(x) \equiv \sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 h(x)}{\partial x_i \partial x_j} \geq 0.$$

Then

$$\int_{R^n} h(x) p_{\Sigma_t}(x) dx \quad ,$$

is an increasing function of t, $0 \leq t \leq 1$.

Proof: Assuming, as before, that $p(\lambda)$ is a $C^2$ function with compact support, then (6) and (8) give

$$\frac{\partial}{dt} \int_{R^n} h(x) p_{\Sigma_t}(x) dx = \int_{R^n} Ah(x) G_{\Sigma_t}(x) dx.$$

By (9) this is non-negative.

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A simple extension of Slepian's inequality is given which implies the original inequality and the result that positively correlated normal variables are associated.

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